University of Vienna
Vienna

MEETING ON APRIL 30th, 1917.

On the Determination of Functions From Their Integral Values Along Certain Manifolds

Author:
Johann Radon

Dr J. Radon

(December 16, 1887 – May 25, 1956)

Berichte über die Verhandlungen der Königlich Sächsischen Gesellschaft der Wissenschaften zu Leipzig

Mathematisch-Physische Klasse
Neunundsechzigster Band

1917

Contents

1 Introduction

When one integrates a function of two variables x, y – a *point function* $f(P)$ in the plane – subject to suitable regularity conditions along an arbitrary straight line g then one obtains in the integral values $F(g)$, a *line function.* In section 2 of the present paper the problem which is solved is the inversion of this linear functional transformation, that is the following questions are answered: can every line function satisfying suitable regularity conditions be regarded as constructed in this way? If so, is f uniquely known from F and how can f be calculated?
In section 3 a solution of the dual problem of calculating a line function $F(g)$ from its point mean values $f(P)$ is solved in a certain sense.
Finally in section 4 certain generalizations are discussed, prompted by consideration of non-Euclidean manifolds as well as higher dimensional spaces. The treatment of these problems, themselves of interest, gains enhanced importance through the numerous relationships that exist between this topic and the theory of logarithmic and NEWTONian potentials. These will be mentioned at appropriate places in the text.

2 Determination of a Point Function in the Plane from its Straight Line Integral Values

1. For all real points $P = [x, y]$ let $f(x, y)$ be a real function satisfying the following regularity conditions:

(a_1) $f(x, y)$ is continuous.

(b_1) The double integral

$$\iint \frac{|f(x,y)|}{\sqrt{x^2+y^2}}\,\mathrm{d}x\,\mathrm{d}y\,.$$

extending over the whole plane, converges.

(c_1) For an arbitrary point $P = [x, y]$ and each $r \geq 0$ let

$$\bar{f}_P(r) = \frac{1}{2\pi}\cdot\int_0^{2\pi} f\left(x + r\cdot\cos(\phi), y + r\cdot\sin(\phi)\right)\mathrm{d}\phi$$

so that for every point P

$$\lim_{r\to\infty} \bar{f}_P(r) = 0\,.$$

Then the following theorems hold good:

Theorem I. The straight line integral value of f along the line g having the equation $x \cdot \cos(\phi) + y \cdot \sin(\phi) = p$ is given by

$$\begin{aligned} F(p,\phi) &= F(-p,\phi+\pi) \\ &= \int_{-\infty}^{+\infty} f\left(p\cdot\cos(\phi) - s\cdot\sin(\phi), p\cdot\sin(\phi) + s\cdot\cos(\phi)\right) \mathrm{d}s \end{aligned} \tag{I}$$

and exists almost everywhere: this means that on every circle the set of tangency points of all tangents for which F does not exist has a linear measure of zero.. □

Theorem II. Constructing the mean value of $F(p,\phi)$ for the tangents of the circle with centre $P = [x, y]$ and the radius q as:

$$\overline{F}_P(q) = \frac{1}{2\pi} \cdot \int_0^{2\pi} F\left(x\cdot\cos(\phi) + y\cdot\sin(\phi) + q, \phi\right) \mathrm{d}\phi \tag{II}$$

then this integral converges absolutely for all P, q. □

Theorem III. The value of f is uniquely determined from F and can be calculated as:

$$f(P) = -\frac{1}{\pi} \cdot \int_0^{\infty} \frac{\mathrm{d}\overline{F}_P(q)}{q} \,. \tag{III}$$

This integral is understood in the sense of a Stieltjes *integral and can also be defined by the equation:*

$$f(P) = \frac{1}{\pi} \cdot \lim_{\varepsilon\to 0} \left(\frac{\overline{F}_P(\varepsilon)}{\varepsilon} - \int_{\varepsilon}^{\infty} \frac{\overline{F}_P(q)}{q^2} \,\mathrm{d}q \right) \,. \tag{III'}$$

□

Before proceeding with the proof we remark that the conditions item (a_1) to item (c_1) are invariant in the face of coordinate shifts in the plane. We can therefore always consider the origin $[0, 0]$ as representing any point of the plane.
One recognises now the double integral:

$$\iint_{x^2+y^2>q^2} \frac{f(x,y)}{\sqrt{x^2+y^2-q^2}} \,\mathrm{d}x\,\mathrm{d}y \tag{1}$$

as absolutely convergent. Through the transformation

$$\begin{aligned} x &= q\cdot\cos(\phi) - s\cdot\sin(\phi) \,, \\ y &= q\cdot\sin(\phi) + s\cdot\cos(\phi) \,, \end{aligned}$$

it becomes

$$\begin{aligned}&\int_0^{2\pi} \mathrm{d}\phi \cdot \int_0^{\infty} f\left(q\cdot\cos(\phi) - s\cdot\sin(\phi), q\cdot\sin(\phi) + s\cdot\cos(\phi)\right) \mathrm{d}s \\ &= \int_0^{2\pi} \mathrm{d}\phi \cdot \int_{-\infty}^{0} f\left(q\cdot\cos(\phi) - s\cdot\sin(\phi), q\cdot\sin(\phi) + s\cdot\cos(\phi)\right) \mathrm{d}s\,,\end{aligned}$$

so that one can express its value as

$$\begin{aligned}&\frac{1}{2}\cdot\int_0^{2\pi} \mathrm{d}\phi \cdot \int_{-\infty}^{+\infty} f\left(q\cdot\cos(\phi) - s\cdot\sin(\phi), q\cdot\sin(\phi) + s\cdot\cos(\phi)\right) \mathrm{d}s \\ &= \frac{1}{2}\cdot\int_0^{2\pi} F(q,\phi)\,\mathrm{d}\phi = \pi\cdot\overline{F}_0(q)\,.\end{aligned}$$

The assertions of Theorems I and II follow from known properties of absolutely convergent double integrals.
To arrive at the Equation III one can take the following course.
Substitution of polar coordinates in Equation 1 above yields

$$\int_q^{\infty} r\,\mathrm{d}r \int_0^{2\pi} \frac{f\left(r\cdot\cos(\phi), r\cdot\sin(\phi)\right)}{\sqrt{r^2-q^2}}\,\mathrm{d}\phi$$

or with the assistance of the mean value expression of item (c_1):

$$2\pi\cdot\int_q^{\infty}\frac{\bar{f}_0(r)r\,\mathrm{d}r}{\sqrt{r^2-q^2}}\,.$$

Comparison with the earlier form of Equation 1 gives:

$$\overline{F}_0(q) = 2\cdot\int_q^{\infty}\frac{\bar{f}_0(r)r\,\mathrm{d}r}{\sqrt{r^2-q^2}}\,. \tag{2}$$

Substituting the variables $r^2 = v$, $q^2 = u$ into this integral equation of the first kind one can easily solve it through the known method of Abel to give the Equation III for

$$\bar{f}_0(0) = f(0,0)\,.$$

This approach appears difficult, however, without further conditions on f, so we prefer a direct verification. First, in order to demonstrate the equality of Equation III and Equation III' one must prove that:

$$\lim_{q\to\infty}\frac{\overline{F}_0(q)}{q} = 0\,.$$

From Equation 2 above we have

$$
\begin{aligned}
\left|\frac{\overline{F}_0(q)}{q}\right| &\leq \frac{2}{q}\cdot\left|\int_q^{2q}\frac{\bar{f}_0(r)r\,\mathrm{d}r}{\sqrt{r^2-q^2}}\right| + \frac{2}{q}\cdot\left|\int_{2q}^{\infty}\frac{\bar{f}_0(r)r\,\mathrm{d}r}{\sqrt{r^2-q^2}}\right| \\
&\leq \frac{2}{q}\cdot\int_q^{2q}\frac{\left|\bar{f}_0(r)\right|r}{\sqrt{r^2-q^2}}\,\mathrm{d}r + \frac{2}{q}\cdot\int_{2q}^{\infty}\frac{\left|\bar{f}_0(r)\right|r}{\sqrt{r^2-q^2}}\,\mathrm{d}r \\
&\leq 2\sqrt{3}\cdot\left|\bar{f}_0(t)\right| + \frac{4}{\sqrt{3}q}\cdot\int_{2q}^{\infty}\left|\bar{f}_0(r)\right|\mathrm{d}r \quad (q\leq t\leq 2q)
\end{aligned}
$$

and this converges to zero as $q\to\infty$ on account of item (b_1) and item (c_1). Through the introduction of Equation 2 the right-hand side of Equation III' now becomes:

$$
\frac{2}{\pi}\cdot\lim_{\varepsilon\to 0}\left[\frac{1}{\varepsilon}\cdot\int_{\varepsilon}^{\infty}\frac{r\bar{f}_0(r)}{\sqrt{r^2-\varepsilon^2}}\,\mathrm{d}r - \int_{\varepsilon}^{\infty}\frac{\mathrm{d}q}{q^2}\cdot\int_q^{\infty}\frac{r\bar{f}_0(r)}{\sqrt{r^2-q^2}}\,\mathrm{d}r\right].
$$

By changing the order of integration in the second integral one can integrate with respect to q, recognizing the integral as an absolutely convergent double integral which justifies this change. One obtains for the whole expression above the equation

$$
\frac{2}{\pi}\cdot\lim_{\varepsilon\to 0}\varepsilon\cdot\int_{\varepsilon}^{\infty}\frac{\bar{f}_0(r)}{r\cdot\sqrt{r^2-\varepsilon^2}}\,\mathrm{d}r\,,
$$

which indeed yields on taking the limit the value $\bar{f}_0(0)=f(0,0)$, as is not difficult to show.

2. Let $F(p,\phi)=F(-p,\phi+\pi)$ be a line function satisfying the following regularity conditions:

(a_2) F and the derivatives F_p, F_{pp}, F_{ppp}, F_ϕ, $F_{p\phi}$ and $F_{pp\phi}$ exist and are continuous for all $[p,\phi]$.

(b_2) F, F_ϕ, pF_p, $pF_{p\phi}$ and pF_{pp} tend to zero as $p\to\infty$ uniformly in ϕ.

(c_2) The integrals:

$$
\int_0^{\infty}F_{pp}\ln(p)\,\mathrm{d}p\,,\quad \int_0^{\infty}F_{ppp}p\ln(p)\,\mathrm{d}p\,,\quad \int_0^{\infty}F_{pp\phi}p\ln(p)\,\mathrm{d}p
$$

converge absolutely and uniformly in ϕ.

Then we can prove:

Theorem IV. Construct $f(P)$ from Equation III or Equation III', thus satisfying the conditions item (a_1), item (b_1), item (c_1), and yielding as the straight line integral value the above function $F(p,\phi)$. As a consequence of Theorem III it is the unique function of this kind.

□

Substituting polar coordinates in Equation III gives

$$\begin{aligned} f\left(\rho\cdot\cos(\psi),\rho\cdot\sin(\psi)\right) &= -\frac{1}{2\pi^2}\cdot\int_0^\infty \frac{\mathrm{d}p}{p}\cdot\int_0^{2\pi} F_p\left(\rho\cdot\cos(\omega)+p,\omega+\psi\right)\mathrm{d}\omega \\ &= \frac{1}{2\pi^2}\cdot\int_0^\infty \ln(p)\,\mathrm{d}p\cdot\int_0^{2\pi} F_{pp}\left(p+\rho\cdot\cos(\omega),\omega+\psi\right)\mathrm{d}\omega\,. \end{aligned}$$

Now

$$\begin{aligned} \int_0^{2\pi} F_p\left(\rho\cdot\cos(\omega)+p,\omega+\psi\right)\mathrm{d}\omega &= \int_0^{2\pi} F_p\left(\rho\cdot\cos(\omega),\omega+\psi\right)\mathrm{d}\omega \\ &\quad+\int_0^{2\pi}\mathrm{d}\omega\cdot\int_0^p F_{pp}\left(\rho\cdot\cos(\omega)+t,\omega+\psi\right)\mathrm{d}t \end{aligned} \tag{IV}$$

and the first component is zero as $F(p,\phi)=F(-p,\phi+\pi)$. For this reason the product of the integral with $\ln(p)$ tends to zero as $p\to 0$. On account of the same property of F it follows also that:

$$\begin{aligned} f\left(\rho\cdot\cos(\psi),\rho\cdot\sin(\psi)\right) = \frac{1}{2\pi^2}\cdot\int_0^\pi \mathrm{d}\omega\cdot\int_{-\infty}^\infty F_{pp}(p,\omega+\psi) \\ \cdot\ln\left|p-\rho\cdot\cos(\omega)\right|\mathrm{d}p\,. \end{aligned} \tag{3}$$

It is sufficient that we show

$$\int_{-\infty}^{+\infty} f(\rho,0)\,\mathrm{d}\rho = F\left(0,\frac{\pi}{2}\right)\,, \tag{4}$$

as the conditions item (a_2) to item (b_2) are invariant in the face of coordinate changes.
We put:

$$F(p,\phi)=F\left(p,\frac{\pi}{2}\right)+\cos(\phi)\cdot G(p,\phi)\,.$$

G satisfies obvious regularity conditions. It is now necessary, on account of this decomposition, to split $f(\rho,0)$ into two parts $f_1(\rho)$ and $f_2(\rho)$, which are investigated

separately. Because

$$\int_0^\pi \ln|p-\rho\cdot\cos(\omega)|\,\mathrm{d}\omega = \begin{cases} \pi\cdot\ln\left\{\frac{|p|+\sqrt{p^2-\rho^2}}{2}\right\}, & \text{wenn } |p|>|\rho| \\ \pi\cdot\ln\left(\frac{|\rho|}{2}\right), & \text{wenn } |p|\le|\rho| \end{cases}$$

one obtains

$$\begin{aligned} f_1(\rho) &= \frac{1}{2\pi^2}\cdot\int_0^\pi \mathrm{d}\omega\cdot\int_{-\infty}^{+\infty} F_{pp}\left(p,\frac{\pi}{2}\right)\cdot\ln|p-\rho\cdot\cos(\omega)|\,\mathrm{d}p \\ &= \frac{1}{2\pi}\cdot\int_{|\rho|}^{\infty} F_{pp}\left(p,\frac{\pi}{2}\right)\cdot\ln\left\{\frac{|p|+\sqrt{p^2-\rho^2}}{|\rho|}\right\}\mathrm{d}p \\ &+ \frac{1}{2\pi}\cdot\int_{-\infty}^{-|\rho|} F_{pp}\left(p,\frac{\pi}{2}\right)\cdot\ln\left\{\frac{|p|+\sqrt{p^2-\rho^2}}{|\rho|}\right\}\mathrm{d}p\,. \end{aligned}$$

This is now absolutely integrable with respect to ρ from $-\infty$ to $+\infty$, as will be obvious through exchanging the order of integration. One evaluates the integral as

$$\begin{aligned} \int_{-\infty}^{+\infty} f_1(\rho)\,\mathrm{d}\rho &= \frac{1}{2\pi}\cdot\int_{-\infty}^{\infty} F_{pp}\left(p,\frac{\pi}{2}\right) \\ &\cdot\int_{-|p|}^{+|p|}\ln\left\{\frac{|p|+\sqrt{p^2-\rho^2}}{|\rho|}\right\}\mathrm{d}\rho\,\mathrm{d}p \\ &= \frac{1}{2}\cdot\int_{-\infty}^{+\infty} F_{pp}\left(p,\frac{\pi}{2}\right)|p|\,\mathrm{d}p = F\left(0,\frac{\pi}{2}\right)\,. \end{aligned}$$

As far as $f_2(\rho)$ is concerned, as we shall show, it is also absolutely integrable and when integrated from $-\infty$ to $+\infty$ gives zero.
We can of course write $f_2(\rho)$ in the following way:

$$\begin{aligned} f_2(\rho) &= \frac{1}{2\pi^2}\cdot\int_0^\pi \mathrm{d}\omega\cdot\int_{-\infty}^{+\infty} G_{pp}(p,\omega)\cdot\ln|p-\rho\cdot\cos(\omega)|\cdot\cos(\omega)\,\mathrm{d}\omega \\ &= \frac{1}{2\pi^2}\cdot\int_0^\pi \mathrm{d}\omega\cdot\int_{-\infty}^{+\infty} G_{pp}(p,\omega)\cdot\left[\ln\left|\frac{p-\rho\cdot\cos(\omega)}{\rho\cdot\cos(\omega)}\right|\cdot\cos(\omega)\right. \\ &\left.+\frac{\rho p\cdot\cos^2(\omega)}{(1+\rho^2\cdot\cos^2(\omega))}\right]\mathrm{d}p \end{aligned}$$

since the added terms result in zero when integrated and in this form the integration

with respect to ρ results in an *absolutely* convergent triple integral. It is

$$\int_{-\infty}^{+\infty} \left| \left[\ln \left| \frac{p - \rho \cdot \cos(\omega)}{\rho \cdot \cos(\omega)} \right| \cdot \cos(\omega) + \frac{\rho p \cdot \cos^2(\omega)}{(1 + \rho^2 \cdot \cos^2(\omega))} \right] \right| \mathrm{d}\rho$$
$$= |p| \cdot \int_{-\infty}^{+\infty} \left| \left[\ln \left| 1 - \frac{1}{\tau} \right| + \frac{p^2 \tau}{(1 + p^2 \tau^2)} \right] \right| \mathrm{d}\tau = \lambda(p)$$

where

$$|p|\tau = \rho \cdot \cos(\omega)$$

with

$$\lim_{|p| \to \infty} \frac{\lambda(p)}{|p| \cdot \ln(|p|)} = 2 \,.$$

The integration with respect to ρ yields as the value of the integral

$$\int_{-\infty}^{+\infty} f_2(\rho) \, \mathrm{d}\rho = 0$$

whereby Equation 4 is proved.
We still have to show that f satisfies the conditions item (a_1) to item (c_1).
The continuity follows from the representation according to Equation 3 on account of assumptions item (a_2) to item (c_2). Condition item (b_1) is equally satisfied because

$$\int_{-\infty}^{+\infty} |f\left(\rho \cdot \cos(\psi), \rho \cdot \sin(\psi)\right)| \, \mathrm{d}\rho$$

is integrable with respect to ψ, as one easily sees.
To prove item (c_1) we set

$$\begin{aligned}
\bar{f}_0(\rho) &= \frac{1}{2\pi} \cdot \int_0^{2\pi} f\left(\rho \cdot \cos(\psi), \rho \cdot \sin(\psi)\right) \mathrm{d}\psi \\
&= \frac{1}{4\pi^2} \cdot \int_0^{\pi} \mathrm{d}\omega \cdot \int_0^{2\pi} \mathrm{d}\psi \cdot \int_{-\infty}^{+\infty} F_{pp}(p, \psi) \cdot \ln |p - \rho \cdot \cos(\omega)| \, \mathrm{d}p \\
&= \frac{1}{4\pi^2} \cdot \int_0^{2\pi} \mathrm{d}\psi \cdot \left\{ \int_{-\infty}^{-|\rho|} F_{pp}(p, \psi) \cdot \ln \left[\frac{|p| + \sqrt{p^2 - \rho^2}}{2} \right] \mathrm{d}p \right. \\
&+ \int_{|\rho|}^{+\infty} F_{pp}(p, \psi) \cdot \ln \left[\frac{|p| + \sqrt{p^2 - \rho^2}}{2} \right] \mathrm{d}p \\
&\left. F_p(\rho, \psi) \cdot \ln \left(\frac{\rho}{2} \right) - F_p(\rho, \psi) \cdot \ln \left(\frac{\rho}{2} \right) \right\} \,,
\end{aligned}$$

from which one can recognize the correctness of item (c_1). Thus, Theorem IV is proved.

3 Determination of a Line Function from its Point Mean Values

3. Let $F(p, \phi) = F(-p, \phi + \pi)$ be a line function that satisfies the following regularity conditions:

(a$_3$) F, F_ϕ, F_p are continuous with $|F_\phi| < M$ for all p, ϕ.

(b$_3$) $F_p \cdot \ln(|p|)$ converges to zero uniformly in ϕ as $p \to \infty$.

(c$_3$) $\int_{-\infty}^{+\infty} |F_p| \cdot \ln(|p|) \, \mathrm{d}p$ converges uniformly in ϕ.

These conditions are moreover invariant with respect to coordinate shifts. We construct the point mean value of $F(p, \phi)$ for $P = [x, y]$:

$$f(x, y) = \frac{1}{\pi} \cdot \int_{-\pi/2}^{+\pi/2} F\left(x \cdot \cos(\phi) + y \cdot \sin(\phi), \phi\right) \mathrm{d}\phi \,. \tag{5}$$

Then the following holds:

Theorem V. Through the expression for f, F is uniquely determined and is given by:

$$F\left(0, \frac{\pi}{2}\right) = -\frac{1}{2\pi} \cdot \int_{-\infty}^{+\infty} \frac{\mathrm{d}x}{x} \cdot \int_{-\infty}^{+\infty} f_x(x, y) \, \mathrm{d}y \,, \tag{V}$$

where the integral with respect to x is understood in the sense of the Cauchy *principal value and the value of F for any other straight line can be derived from the given formula through an appropriate coordinate transformation.* □

For the proof, we next derive from Equation 5:

$$\int_{-A}^{B} f_x(x, y) \, \mathrm{d}y = \frac{1}{\pi} \cdot \int_{-\pi/2}^{+\pi/2} \mathrm{d}\phi \cdot \int_{-A}^{B} F_p\left(x \cdot \cos(\phi) + y \cdot \sin(\phi), \phi\right) \cdot \cos(\phi) \, \mathrm{d}y \,, \tag{6}$$

where A, B are two positives constants.
We now put, as was done analogously earlier on:

$$F(p, \phi) = F(p, 0) + \sin(\phi) \cdot G(p, \phi) \,,$$

where $G(p, \phi)$ remains restricted to the region of integration and takes the limiting

value of zero as $p \to \infty$. From:

$$\begin{aligned}&\int_{-A}^{B} G_p\left(x \cdot \cos(\phi) + y \cdot \sin(\phi), \phi\right) \cdot \cos(\phi) \cdot \sin(\phi)\,\mathrm{d}y \\ &= \left[G\left(x \cdot \cos(\phi) + B \cdot \sin(\phi), \phi\right) - G\left(x \cdot \cos(\phi) - A \cdot \sin(\phi), \phi\right)\right] \cdot \cos(\phi)\end{aligned}$$

it follows that the second component of Equation 6 takes the limiting value of zero for $A \to \infty$, $B \to \infty$, so that it remains only to investigate the first component. Through the analogous integration one recognizes that in this first component the integral with respect to ϕ likewise will be zero for $A \to \infty$, $B \to \infty$ over any interval *not* containing ϕ; it remains therefore to examine

$$\lim_{\substack{A \to \infty \\ B \to \infty}} \int_{-\varepsilon}^{+\varepsilon} \mathrm{d}\phi \cdot \int_{-A}^{B} F_p\left(x \cdot \cos(\phi) + y \cdot \sin(\phi), 0\right) \cdot \cos(\phi)\,\mathrm{d}y\,, \quad 0 < \varepsilon \leq \frac{\pi}{2}$$

One can write this integral as

$$\frac{1}{\pi} \cdot \int_{-\varepsilon}^{+\varepsilon} \mathrm{d}\phi \cdot \int_{x \cdot \cos(\phi) - A \cdot \sin(\phi)}^{x \cdot \cos(\phi) + B \cdot \sin(\phi)} F_p(p, 0) \cdot \cot(\phi)\,\mathrm{d}p$$

and obtain from it, when A and B take sufficiently large values, through exchange of the order of integration and after some calculation, the value

$$\begin{aligned}&\frac{1}{\pi} \cdot \int_{x \cdot \cos(\varepsilon) - B \cdot \sin(\varepsilon)}^{x \cdot \cos(\varepsilon) + B \cdot \sin(\varepsilon)} \ln\left\{\frac{(B^2 + x^2) \cdot \sin(\varepsilon)}{\left|Bp - x \cdot \sqrt{B^2 + x^2 - p^2}\right|}\right\} \cdot F_p(p, 0)\,\mathrm{d}p \\ &+ \frac{1}{\pi} \cdot \int_{x \cdot \cos(\varepsilon) - A \cdot \sin(\varepsilon)}^{x \cdot \cos(\varepsilon) + A \cdot \sin(\varepsilon)} \ln\left\{\frac{(A^2 + x^2) \cdot \sin(\varepsilon)}{\left|Ap - x \cdot \sqrt{A^2 + x^2 - p^2}\right|}\right\} \cdot F_p(p, 0)\,\mathrm{d}p\,.\end{aligned}$$

It is sufficient to determine the limiting value of the second integral for $A \to \infty$. We therefore write it as:

$$\begin{aligned}&\frac{1}{\pi} \cdot \ln\left[A \cdot \sin(\varepsilon)\right] \cdot \left[F\left(x \cdot \cos(\varepsilon) + A \cdot \sin(\varepsilon), 0\right) - F\left(x \cdot \cos(\varepsilon) - A \cdot \sin(\varepsilon), 0\right)\right] \\ &+ \frac{1}{\pi} \cdot \int_{x \cdot \cos(\varepsilon) - A \cdot \sin(\varepsilon)}^{x \cdot \cos(\varepsilon) + A \cdot \sin(\varepsilon)} \ln\left\{\frac{1}{|p - x|}\right\} \cdot F_p(p, 0)\,\mathrm{d}p \\ &+ \frac{1}{\pi} \cdot \int_{x \cdot \cos(\varepsilon) - A \cdot \sin(\varepsilon)}^{x \cdot \cos(\varepsilon) + A \cdot \sin(\varepsilon)} \ln\left\{\frac{\left|Ap + x \cdot \sqrt{A^2 + x^2 - p^2}\right|}{A \cdot |p + x|}\right\} \cdot F_p(p, 0)\,\mathrm{d}p\,.\end{aligned}$$

Since the logarithm in the last integral tends *uniformly* to zero for $A \to \infty$ then

the limiting value follows as:

$$-\frac{1}{\pi} \cdot \int_{-\infty}^{+\infty} F_p(p,0) \cdot \ln|p-x| \, \mathrm{d}p \,,$$

whereby the limiting value of Equation 6 is obtained as

$$\int_{-\infty}^{+\infty} f_x(x,y) \, \mathrm{d}y = -\frac{2}{\pi} \cdot \int_{-\infty}^{+\infty} F_p(p,0) \cdot \ln|p-x| \, \mathrm{d}p \,.$$

It should be noted here that the latter expression represents the limiting values of the imaginary part of an analytic function which is regular in the upper plane and for which the limiting values of the real part have the value $2F(x,0)$.
We now form in the sense of Equation V in Theorem V:

$$-\int_{-\infty}^{+\infty} \frac{\mathrm{d}x}{x} \cdot \int_{-\infty}^{+\infty} f_x(x,y) \, \mathrm{d}y = \frac{2}{\pi} \cdot \int_{0}^{\infty} \frac{\mathrm{d}x}{x} \cdot \int_{-\infty}^{+\infty} F_p(p,0) \cdot \ln\left|\frac{p-x}{p+x}\right| \mathrm{d}x \,,$$

so that this double integral is absolutely convergent and because

$$\int_0^{\infty} \ln\left|\frac{p-x}{p+x}\right| \cdot \frac{\mathrm{d}x}{x} = -\frac{\pi^2}{2} \cdot \operatorname{sgn} p$$

it leads precisely to Equation V in Theorem V.

4. Now let f be a point function with the following regularity properties:

(a_4) f including its derivatives of up to the second order is continuous.

(b_4) The expressions

$$\begin{aligned}
&f(x,y) \\
&\sqrt{x^2+y^2} \cdot \ln\left(x^2+y^2\right) \cdot f_x(x,y) \\
&\sqrt{x^2+y^2} \cdot \ln\left(x^2+y^2\right) \cdot f_y(x,y)
\end{aligned}$$

have the limiting value zero for $x^2+y^2 \to \infty$.

(c_4) The integrals

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} D_1 f \cdot \frac{\ln\left(x^2+y^2\right)}{\sqrt{x^2+y^2}} \, \mathrm{d}x \, \mathrm{d}y$$

$$\text{and} \quad \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} D_2 f \cdot \ln\left(x^2+y^2\right) \mathrm{d}x \, \mathrm{d}y \,,$$

where $D_1 f$ represents every first derivative of f and $D_2 f$ every second derivative, all converge absolutely.

These conditions are again invariant with coordinate changes.

Then the following holds:

Theorem VI. The straight line function calculated from f according to Equation V takes the point mean values $f(x, y)$. □

It is sufficient to furnish the proof for the origin. For an arbitrary straight line through it, Equation V yields, following a partial integration

$$F(0,\phi) = \frac{1}{2\pi} \cdot \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \big\{ f_{xx} \cdot \cos^2(\phi) + 2 f_{xy} \cdot \sin(\phi)\cos(\phi) \\ + f_{yy} \cdot \sin^2(\phi) \big\} \cdot \ln |x \cdot \cos(\phi) + y \cdot \sin(\phi)| \, \mathrm{d}x \, \mathrm{d}y$$

or, introducing polar coordinates ρ, ψ:

$$\begin{aligned} F(0,\phi) = \frac{1}{2\pi} \cdot \int_0^{\infty} \rho \, \mathrm{d}\rho \cdot \int_0^{2\pi} & \left\{ \frac{\partial^2 f}{\partial \rho^2} \cdot \cos^2(\phi - \psi) \right. \\ & + 2 \cdot \frac{\partial^2 f}{\partial \rho \, \partial \psi} \cdot \frac{\sin(\phi-\psi) \cdot \cos(\phi-\psi)}{\rho} \\ & + \frac{\partial^2 f}{\partial \psi^2} \cdot \frac{\sin^2(\phi-\psi)}{\rho^2} + \frac{\partial f}{\partial \rho} \cdot \frac{\sin^2(\phi-\psi)}{\rho} \\ & \left. - 2 \cdot \frac{\partial f}{\partial \psi} \cdot \frac{\sin(\phi-\psi)\cdot\cos(\phi-\psi)}{\rho^2} \right\} \\ & \cdot \ln|\rho \cdot \cos(\phi - \psi)| \, \mathrm{d}\psi \,. \end{aligned}$$

In order to construct the point mean value for $[0, 0]$ one can carry out the integration with respect to ψ in the double integral from 0 to 2π and then divide by 2π. The term with $\frac{\partial^2 f}{\partial \psi^2}$ that appears drops out in the integration with respect to ψ and so there remains

$$\frac{1}{2\pi} \cdot \int_0^{2\pi} \mathrm{d}\psi \cdot \int_0^{\infty} \left\{ \frac{1}{4} \cdot \left(\rho \cdot \frac{\partial^2 f}{\partial \rho^2} - \frac{\partial f}{\partial \rho} \right) + \frac{1}{2} \cdot \ln\left(\frac{\rho}{2}\right) \cdot \frac{\partial}{\partial \rho} \left(\rho \cdot \frac{\partial f}{\partial \rho} \right) \right\} \mathrm{d}\rho \,,$$

which reduces in fact to $f(0, 0)$.
In order to demonstrate the uniqueness of F, we must prove that Conditions item (a_3) to item (c_3) are satisfied, which apparently requires further conditions on f

5. There should be space here for the following remark, which I owe, as well as the problem stated in section 3, to Mr. W. Blaschke: both problems considered here are closely related to the theory of Newton's potential. Namely, let us consider the transition from a point function f to its straight-line mean values F as a linear functional transformation

$$F = R \cdot f\,,$$

and likewise the conversion of a line function F to its point mean values v:

$$v = B \cdot F\,,$$

suggesting that the compounded, according to

$$v = H \cdot f = B\,[R \cdot f] = BR \cdot f$$

defined transformation $H = BR$ should be considered.
One now sees immediately that $H \cdot f$ is none other than the Newtonian potential of the plane covered with the mass density $\frac{1}{\pi} \cdot f$ calculated at points of the plane itself. It follows that one can obtain the inversion of the transformation H according to a remark by G. Herglotz; this results in

$$f(P) = \frac{v}{H} = -\frac{1}{2} \cdot \int_0^\infty \frac{\mathrm{d}\overline{v}_P(r)}{r} = -\frac{1}{4\pi} \cdot \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{\Delta v(x', y')}{r_{PP'}}\, \mathrm{d}x'\, \mathrm{d}y'\,,$$

where $\overline{v}_p$ is a mean value expression analogous to that introduced earlier and Δ denotes the Laplace operator.
Now the idea is that the inversion of R and H, which is directly guided by the approach in Equation 1 to Equation 3 is achieved through the expressions

$$\frac{1}{R} = \frac{B}{H} \quad \text{bzw.} \quad \frac{1}{B} = \frac{R}{H}\,.$$

Actually I first discovered the inversion formula according to Equation IV in this way, but a rigorous development of this conception appeared to be more difficult than the direct verification and also fails in the non-Euclidean cases to be discussed shortly.
Finally, it should be noted that the basic regularity conditions in section 2 and section 3 are obviously far from the most general, as may be shown by simple examples.

4 Generalizations

6. A far-reaching generalization of the problem treated in section 2 can be formulated as follows: let a surface S be given on which an arc differential $\mathrm{d}s$ is somehow defined as well as a double infinite sheaf of curves C on S. It is required to determine a point function of the surface from its integral values $\int f\,\mathrm{d}s$ along the curves C.
One obtains the nearest specialization when one takes a non-Euclidean plane as S, the appropriate arc element for $\mathrm{d}s$ and straight lines for the curves C. In the elliptical cases one can bring into play the results of spherical geometry; in a known manner one interprets a diametrical point-pair of the sphere as a point of the elliptical plane and with this obtaining the task to determine a direction function on the sphere – that means equal valued at diametric points – from its great circle integral values. MINKOWSKI has been the first to consider this result in principle and to have solved it through the use of the spherical functions[1]; P. FUNK later performed the MINKOWSKI solution and showed how one can find the solution with the help of ABEL'ian integral equations[2]. I also am indebted to this method for the solution of the Problem given in section 2. FUNK's solution is altogether analogous to Equation III, only substituting in the denominator the sine of the spherical radius and adding to the integral the value of F at the pole of the great circle in question divided by π. The stated result also has an analogous solution to Equation III in the hyperbolic plane:

$$f(P) = -\frac{1}{\pi} \cdot \int_0^\infty \frac{\mathrm{d}\overline{F}_P(q)}{\sinh(q)}$$

(here the curvature measure is taken as $= |1)$, as one can show totally conforms to the derivation of Equation III on page 3.
In both cases, one can also for example pose the problem analogous to section 3. In the elliptical geometry one obtains nothing new by virtue of the absolute polarity: in the hyperbolic case a solution analogous to Equation V in Theorem V appears not to exist.
A second specialization follows when one takes as the curves C circles with constant radius (in EUCLIDean or non-EUCLIDean geometries). Here one can make use of the MINKOWSKI procedure using spherical functions on the sphere and solve the problem to a certain extent. It is interesting in this case that the uniqueness of the solution can be lost; For certain radii ρ defined by the zeros of the LEGENDRE polynomials of even order, there are even functions on the sphere which, when integrated along each circle of the spherical radius ρ, result in zero, without vanishing identically. In the Euclidean case the integral theorem of BESSEL functions takes the place of the spherical function sequence; here there are always functions that integrate on

[1] Ges. Abh., Bd. II, S. 277f.
[2] Math. Ann., Bd. 74, S. 283–288.

all circles of constant radius giving zero and still do not vanish identically; if the radius is in unity then (in polar coordinates ρ, ϕ) these functions are

$$J_n(x_\nu \rho) \cdot \cos(n\phi)\,, \quad J_n(x_\nu \rho) \cdot \sin(n\phi)\,,$$

and their linear combinations where x_ν is a zero of J_0. In hyperbolic cases the so-called spherical functions take the place of the BESSEL functions for which the appropriate integral theorem of WEYL[3] is proved. The results are analogous to the EUCLIDean cases.

7. In another direction the results of section 2 and section 3 can be generalized by transposition to higher dimensions. In an EUCLIDean $\mathbb{R}^n$ one can seek to determine a point function $f(P) = f(x_1, x_2, \ldots, x_n)$ from its integral values $F(\alpha_1, \alpha_2, \ldots, \alpha_n, p)$ on all hyperplanes $\alpha_1 x_1 + \cdots + \alpha_n x_n = p \cdot (\alpha_1^2 + \cdots + \alpha_n^2 = 1)$. Analogous to the procedure followed on page 3 we construct the mean value $\overline{F}_0(q)$ of F on the tangent planes of the sphere from the centre of $[0, 0, \ldots, 0]$ and radius q. It is given by the $n-1$-fold integral:

$$\overline{F}_0(q) = \frac{1}{\Omega_n} \cdot \int F(\alpha, q)\,\mathrm{d}\omega$$

where $\mathrm{d}\omega$ is the surface element, $\Omega_n = \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}$ the surface area of the n-dimensional sphere $x_1^2 + \cdots + x_n^2 = 1$.
One can describe $\overline{F}_0$ through an n-fold integral in f and indeed this is

$$\overline{F}_0(q) = \frac{\Omega_{n-1}}{\Omega_n} \cdot \int \cdots \int_{x_1^2 + \cdots + x_n^2 > q^2} f(x_1, x_2, \ldots, x_n) \cdot \frac{(x_1^2 + \cdots x_n^2 - q^2)^{\frac{(n-3)}{2}}}{(x_1^2 + \cdots + x_n^2)^{\frac{(n-2)}{2}}}\,\mathrm{d}x_1 \cdots \mathrm{d}x_n\,, \tag{7}$$

or in a frequently used mean value expression

$$\overline{F}_0(q) = \Omega_{n-1} \cdot \int_q^\infty \bar{f}_0(r) \cdot \left(r^2 - q^2\right)^{\frac{(n-3)}{2}} \cdot r\,\mathrm{d}r\,.$$

This is the formula analogous to Equation 2 to which are connected the corresponding conclusions. The substitution $r^2 = v$, $q^2 = u$ leads to the integral equation

$$\Phi(u) = \frac{\Omega_{n-1}}{2} \cdot \int_u^\infty \phi(v) \cdot (v-u)^{\frac{(n-3)}{2}}\,\mathrm{d}v\,.$$

If n is *even*, $\left(\frac{n}{2} - 1\right)$ differentiations with respect to u gives the same equation as

[3] Gött. Nachr. 1910, S. 454

Equation 2 and one can find from this

$$\phi(0) = f(0, 0, \ldots, 0)\,.$$

Therefore, for the construction of f from a given F, both differentiation and an integral operation are necessary. For *odd* n this integral operation becomes redundant because $\left(\frac{n-1}{2}\right)$-fold differentiation now gives

$$\phi(0) = \frac{2 \cdot (-1)^{\frac{(n-1)}{2}}}{\Omega_{n-1} \cdot \left(\frac{(n-3)}{2}\right)!} \cdot \Phi^{\frac{(n-3)}{2}}(0)\,.$$

The three-dimensional case turns out to be particularly simple; one can treat this case also by a method that is analogous to 5., page 14 and which yields very elegant results. From Equation 7 the point mean value of F for $q = 0$ emerges as

$$\overline{F}_0 = \frac{1}{2} \cdot \iiint \frac{f(x, y, z)}{\sqrt{x^2 + y^2 + z^2}}\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z\,,$$

which is considered as the NEWTONian potential of the space filled with a mass density $\frac{1}{2}f$. Hence, it follows that

$$f(x, y, z) = -\frac{1}{2\pi} \cdot \Delta\overline{F}$$

where $\overline{F}$ indicates the point mean value of F.
Here one can also solve the problem which is analogous to section 3 and obtain by the methods indicated in 5., page 14, for planar functions F of which the point mean values f are known

$$F(E) = -\frac{1}{2\pi} \cdot \iint \Delta f\, \mathrm{d}\sigma\,,$$

where $\mathrm{d}\sigma$ the surface element of the plane E. Δ is the LAPLACE operator for three-dimensional space and the integration is extended over the whole plane E.

Declared ready for printing: October 1, 1917.